

A POINCARÉ-BIRKHOFF-WITT THEOREM FOR GENERALIZED LIE COLOR ALGEBRAS

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ABSTRACT. A proof of Poincaré-Birkhoff-Witt theorem is given for a class of generalized Lie algebras closely related to the Gurevich S -Lie algebras. As concrete examples, we construct the positive (negative) parts of the quantized universal enveloping algebras of type A_n and $M_{p,q,\epsilon}(n, \mathbb{K})$, which is a non-standard quantum deformation of $GL(n)$. In particular, we get, for both algebras, a unified proof of the Poincaré-Birkhoff-Witt theorem and we show that they are genuine universal enveloping algebras of certain generalized Lie algebras.

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Running title: PBW for generalized Lie color algebras

I. INTRODUCTION

In the paper [1], H. Yamane presented a proof of the Poincaré-Birkhoff-Witt (PBW) theorem for some class of quantum groups: Drinfeld-Jimbo quantum groups of type A_n . In his proof he did not use explicitly the Lie algebra theory concepts.

In this paper we show that Yamane used in an implicit manner some generalized Lie algebra. Such a generalized Lie algebra will be called T -Lie algebra.

The T -Lie algebras satisfy not only generalized antisymmetry and Jacobi identity, but additional properties like multiplicativity, (also generalized, in the same way as the Gurevich S -Lie algebras [2]). Such T -Lie algebras arise in a natural way embedded in the positive and negative parts of the Drinfeld-Jimbo quantum groups $U_q(sl_{n+1})$ of type A_n .

Our T -Lie algebras share some properties with the S -Lie algebras. But they are not equivalent, for example, T -Lie algebras satisfy a weaker multiplicativity condition. In particular, there are some T -Lie algebras which are not S -Lie algebras. However, classical Lie algebras [3], Lie superalgebras [4] and Scheunert generalized Lie algebras (Lie color algebras) [5] are all T -Lie algebras.

These T -Lie algebras are related to the problem of finding the appropriate definition of a *quantum Lie algebra*. There are already some generalized Lie algebras proposed to solve this problem: Majid braided Lie algebras [6], Delius-Gould quantum Lie algebras [7], new generalized Lie algebras of Gurevich-Rubstov [8], generalized Lie algebras due to Lyubashenko-Sudbery [9], among others. But the Delius-Gould definition and the Gurevich-Rubstov also, depends on the associated universal enveloping algebra. This is not the case for the T -Lie algebras. Our axioms imply the properties of the universal enveloping algebra. In particular we shall prove the PBW theorem.

The generalized Lie algebras axioms of Lyubashenko-Sudbery are not enough in order to obtain a PBW theorem (see example IV.4). While the main difference

with the braided Lie algebras of Majid is that the symmetry of our T -Lie algebras is not a braid morphism. Only a part of such symmetry is braided.

In particular, we get a T -Lie algebra $(sl_{n+1}^\pm)_q$ which is a deformation of the Lie subalgebra of upper (lower) triangular matrices. Such generalized Lie algebra meets almost all the requirements of a *quantum Lie algebra* in the sense of Lyubashenko-Sudbery [9], (only fails the point 7; actually the universal enveloping algebra of $(sl_{n+1}^\pm)_q$ has no a Hopf algebra structure, but it seems possible to define a *braided* Hopf algebra on it, however we do not try such matter in this paper). Moreover, the universal enveloping of $(sl_{n+1}^\pm)_q$ is $U_q^\pm(sl_{n+1})$ the positive part of the Drinfeld-Jimbo quantum group of type A_n , therefore the diagram in Figure 1 commutes.

This means that, relative to $U_q^\pm(sl_{n+1})$, the T -Lie algebra $(sl_{n+1}^\pm)_q$ satisfies, in

$$\begin{array}{ccc} sl_{n+1}^\pm & \longrightarrow & (sl_{n+1}^\pm)_q \\ \downarrow & & \downarrow \\ U(sl_{n+1}^\pm) & \longrightarrow & U_q^\pm(sl_{n+1}) = U(sl_{n+1}^\pm)_q \end{array}$$

FIGURE 1. The classical sl_{n+1}^\pm and the quantum $(sl_{n+1}^\pm)_q$

some sense, the quantum Lie algebra condition of Delius [10].

Some possible physical applications of the formalism of generalized Lie algebras are in the affine Toda theories [11], quantum integrable systems [11], and gauge theory [9].

The paper is organized as follows. In Sec. II we shall define the T -Lie algebras. In Sec. III a list of classical and new Lie algebras is given. In Sec. IV we shall define the universal enveloping algebra of a T -Lie algebra and we shall prove that expecting an analogue at PBW theorem for any such universal enveloping algebra constructed by means of commutators is too much, we have to restrict our generalized Lie algebras in an adequate way. However, in Sec. V we persuit the classical idea to prove the PBW theorem [3] by constructing a representation of the universal enveloping algebra on the symmetric algebra (with modifications inspirated by [1]). In Sec. VI the definition of a representation of T -Lie algebra is given. In Sec. VII we shall prove the PBW theorem for the universal enveloping of an adequate T -Lie algebra. Some remarks about braid morphisms are given in Sec. VIII. The Sec. IX is devoted to explain why we can apply the T -Lie algebras theory to a non-standard quantum deformation algebra [12] of $GL(n)$. Similar explanations are given in Sec. X but now dealing with $U_q^\pm(sl_{n+1})$ the positive (negative) parts of the Drinfeld-Jimbo quantum groups of type A_n . In particular, in Sec. X we shall prove that $U_q^\pm(sl_{n+1})$ is a genuine universal enveloping algebra of certain T -Lie algebra.

II. THE NOTION OF T -LIE ALGEBRA

Let k be a commutative unitary ring.

Definition II.1. A k -algebra A is strictly graded if there exist k -submodules $(A_\eta)_{\eta \in \mathbb{N}}$ such that

$$A = \bigoplus_{\eta \in \mathbb{N}} A_\eta \text{ and } A_{\eta_1} \cdot A_{\eta_2} \subseteq A_{\eta_1 + \eta_2 - 1}$$

for all $\eta_1, \eta_2 \in \mathbb{N}$. For $a \in A_\eta$, we shall put $\eta(a) = \eta$.

Remark II.1. For such graded algebras A we can induce a filtration of $A \otimes_k A$ given by

$$(A \otimes_k A)_\eta = \bigoplus_{\eta_1 + \eta_2 \leq \eta} A_{\eta_1} \otimes A_{\eta_2}$$

Let L be a free k -module with a given basis \mathcal{B} totally ordered.

Definition II.2. Denote by L^n the k -submodule of $L^{n\otimes}$ generated by

$$x_{i_1} \otimes \dots \otimes x_{i_n}, \quad x_{i_1} < \dots < x_{i_n}, \quad (x_{i_j} \in \mathcal{B}),$$

and by ${}^n L$ the k -submodule generated by

$$x_{i_1} \otimes \dots \otimes x_{i_n}, \quad x_{i_1} > \dots > x_{i_n}, \quad (x_{i_j} \in \mathcal{B}).$$

Definition II.3. The module L with k -morphisms

$$S : L \otimes_k L \rightarrow L \otimes_k L, \text{ (presymmetry)} \quad (2.1)$$

$$T : L \otimes_k L \rightarrow L \otimes_k L, \text{ (symmetry)} \quad (2.2)$$

$$\langle, \rangle : L \otimes_k L \rightarrow L \otimes_k L, \text{ (pseudobacket)} \quad (2.3)$$

$$[,] : L \otimes_k L \rightarrow L, \text{ (bracket)} \quad (2.4)$$

is called *T-Lie algebra with basis \mathcal{B} (or basic T-Lie algebra)* if, for $S_{12} = S \otimes_k Id_L$, $S_{23} = Id_L \otimes_k S$, the following axioms are satisfied:

1. (a) $S^2 = Id$
 (b) $S(x \otimes y) = q_{x,y} y \otimes x$, for certain $q_{x,y} \in k$, $\forall x, y \in \mathcal{B}$
 (c) (Multiplicativity)
 (i) $S(Id \otimes_k [,])|_{L^3} = ([,] \otimes_k Id) S_{23} S_{12}|_{L^3}$
 (ii) $S([,] \otimes_k Id)|_{L^3} = (Id \otimes_k [,]) S_{12} S_{23}|_{L^3}$
2. (Stability)
 (a) There exists a strict grading

$$L = \bigoplus_{\eta} L_{\eta}$$

of L relative to $[,]$.

(b)

$$\langle L_{\eta_1} \otimes L_{\eta_2} \rangle \subseteq (L \otimes_k L)_{\eta_1 + \eta_2 - 1}$$

for all L_{η_1}, L_{η_2} .

3. $T = S + \langle, \rangle$
4. (Antisymmetry)
 (a) $[,]T = -[,]$
 (b) $\langle, \rangle S = -\langle, \rangle$
 (c) $[,]\langle, \rangle = 0$
5. (Jacobi Identity)

$$[,]((Id \otimes_k [,]) S_{12} S_{23} - ([,] \otimes_k Id) S_{23} S_{12} + (Id \otimes_k [,]) S_{23} S_{12})|_{L^3} = 0$$

Multiplicativity conditions are to control commutation relations in the universal enveloping algebra, whereas stability conditions are to obtain a good gradation in the corresponding symmetric algebra.

Definition II.4. Let L_i be a basic T-Lie algebra with bracket $[_,]_i$, pseudobacket \langle, \rangle_i and presymmetry S_i , $i = 1, 2$. A k -morphism $f : L_1 \rightarrow L_2$ is called a *T-Lie morphism* if f is a morphism of graded algebras relative to $[_,]_i$, $i = 1, 2$ and the diagrams in the Figure 2 commute.

$$\begin{array}{ccccccc}
L_1 \otimes_k L_1 & \xrightarrow{\langle, \rangle_1} & L_1 \otimes_k L_1 & L_1 \otimes_k L_1 & \xrightarrow{S_1} & L_1 \otimes_k L_1 \\
f \otimes f \downarrow & & \downarrow f \otimes f & f \otimes f \downarrow & & \downarrow f \otimes f \\
L_2 \otimes_k L_2 & \xrightarrow{\langle, \rangle_2} & L_2 \otimes_k L_2 & L_2 \otimes_k L_2 & \xrightarrow{S_2} & L_2 \otimes_k L_2
\end{array}$$

FIGURE 2. A T -Lie algebra morphism

III. EXAMPLES

In order to obtain a graduation in the stability conditions it suffices to define a map $\eta : \mathcal{B} \rightarrow \mathbb{N}$ having properties (2a) and (2b) in the stability axiom. This remark will be used in the following examples.

A. Some common Lie algebras.

Example III.1. Classical Lie algebras over fields are basic T -Lie algebras:

$$[,] \text{ classical bracket, } \langle, \rangle = 0, T = S \text{ usual swicht, } \eta = 1.$$

Example III.2. Lie superalgebras over fields [4] are basic T -Lie algebras:

Let $L = L_0 \oplus L_1$ be a Lie superalgebra with bracket $[,]$ over a field k . Let \mathcal{B}_α basis of L_α , $\alpha = 0, 1$. Define $S : L \otimes_k L \rightarrow L \otimes_k L$ on the basis $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_1$, by $S(x \otimes y) = (-1)^{\alpha\beta} y \otimes x$ if $x \in L_\alpha$, $y \in L_\beta$. Besides

$$\langle, \rangle = 0, T = S, \eta = 1.$$

Example III.3 (Lie color algebras). Let k be a field of characteristic zero. Let

$$L = \bigoplus_{\gamma \in \Gamma} L_\gamma$$

be a ϵ Lie algebra [5], where Γ is an abelian group and ϵ is a commutation factor on Γ . Let $[,]$ be bracket of L . Put \mathcal{B}_γ basis of L_γ for each $\gamma \in \Gamma$.

Define

$$S(x \otimes y) = \epsilon(\alpha, \beta) y \otimes x, \text{ if } x \in \mathcal{B}_\alpha, y \in \mathcal{B}_\beta,$$

besides $\langle, \rangle = 0$ and $\eta = 1$.

Multiplicativity conditions follow easily from the definition of commutation factor. We conclude that every ϵ Lie algebra is a T -Lie algebra.

B. Linear T -Lie algebras.

Example III.4. Let e_{ij} , $1 \leq i, j \leq n$ be standard basis of gl_n matrices $n \times n$ over a field \mathbb{K} . Let $[,]$ be the usual bracket in gl_n , sl_n^+ the Lie subalgebra of upper triangular matrices having trace zero. We put $x_i = e_{i, i+1}$, $i = 1, \dots, n-1$, $x_n = [x_1, x_2]$, $x_{n+1} = [x_2, x_3]$, \dots , $x_{2n-3} = [x_{n-2}, x_{n-1}]$, $x_{2n-2} = [x_1, x_{n+1}] \dots$, $x_{3n-6} = [x_{n-3}, x_{2n-3}]$, $x_{3n-5} = [x_1, x_{2n-1}]$, $x_{3n-4} = [x_2, x_{2n}]$, \dots , $x_m = [x_1, x_{m-1}]$ where $m = n(n-1)/2$, besides we define $h_i = [x_i, x_i^t]$, $i = 1, \dots, m$ diagonal matrices in sl_n . Further, $q = \exp(t) \in \mathbb{K}[[t]]$ formal series ring with indeterminate t and coefficients in \mathbb{K} , $k = \mathbb{K}[q, q^{-1}]$, $c_{i,j} \in \mathbb{Z}$ such that $[h_i, x_j] = c_{i,j} x_j$, $1 \leq i, j \leq m$.

Let $(sl_n^+)_q$ be a free k -module with basis $\mathcal{B} = \{x_i \mid 1 \leq i \leq m\}$. We may define an order in \mathcal{B} according to the Figure 3, from left to right and up to bottom. For example $x_1 < x_n < x_2 < x_{2n}$. The first time that a diagram (Auslander-Reiten quiver of type A_{n-1}) of this type appears related to quantum groups, is in Ringel's

work about the relationship between Poincaré-Birkhoff-Witt bases, quantum groups and Hall algebras [13].

Define:

$$[x, y]_q = [x, y] \text{ if } x < y \in \mathcal{B}$$

$$\langle e_{ij}, e_{uv} \rangle = \begin{cases} (q - q^{-1})e_{iv} \otimes e_{uj} & \text{if } i < u < j, u < j < v, \\ (q^{-1} - q)e_{uj} \otimes e_{iv} & \text{if } u < i < v, i < v < j \\ 0, & \text{otherwise.} \end{cases}$$

$$S(x_i \otimes x_j) = q^{c_{i,j}} x_j \otimes x_i, \text{ if } x_i < x_j,$$

$$T = S + \langle, \rangle$$

Finally, we define η in such way that every basic element in the Figure 3 is in correspondence with a number belonging to the Figure 4, this yields, $\eta(e_{ij}) = i(j - i)$, $\forall i, j$.

The multiplicativity condition follows from properties

$$x_i < x_j < x_l \Rightarrow \begin{cases} [x_i, x_j] < x_l, \text{ if } [x_i, x_j] \neq 0, \\ x_i < [x_j, x_l], \text{ if } [x_j, x_l] \neq 0, \end{cases}$$

and

$$[h_i, [x_j, x_l]] = (c_{i,j} + c_{i,l})[x_j, x_l]$$

In the cases $n = 2, 3, 4, 5$, the Jacobi identity for $[\cdot, \cdot]_q$ can be verified by straightforward calculations. We get that $(sl_n^+)_q$ with bracket $[\cdot, \cdot]_q$ is a basic T -Lie algebra, $n = 2, 3, 4, 5$.

Similarly, we can define $(sl_n^-)_q$.

Remark III.1. We get

$$(sl_n^\pm)_q|_{t=0} = sl_n^\pm,$$

so in the cases $n = 2, 3, 4, 5$, $(sl_n^\pm)_q$ is a deformation of sl_n^\pm in the category of T -Lie algebras. Later, in the section X, such property will be generalized for every n .

Example III.5. Starting from $(sl_4^+)_q$ we are going to build a new basic T -Lie algebra, denoted $(\widetilde{sl_4^+})_q$. Its structure is:

$$[\widetilde{\cdot}]_q = [\cdot]_q, \quad \langle \widetilde{\cdot} \rangle = 0, \quad \widetilde{S} = S, \quad \widetilde{T} = S, \quad \widetilde{\eta} = \eta.$$

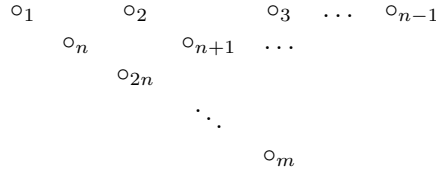


FIGURE 3. The basic T -Lie algebra $(sl_{n+1}^+)_q$

$$\begin{array}{cccccc}
1 & & 2 & & 3 & \dots & n \\
& & 2 & & 4 & & \dots & 2(n-1) \\
& & & & 3 & & 6 & \dots & 3(n-2) \\
& & & & & & \ddots & & \\
& & & & & & & & n
\end{array}$$

FIGURE 4. The graduation of $(sl_{n+1}^+)_q$

Example III.6 (Non-standard quantum deformations [12] of $GL(n)$). Let p, q be units in a commutative unitary ring k with $pq \neq 1$ and choose $n(n-1)/2$ discrete parameters ϵ_{ij} , $\epsilon_{ij} = \pm 1$, $1 \leq i < j \leq n$, $\epsilon_{ii} = 1$, $\epsilon_{ji} = \epsilon_{ij}$.

The k -module $L_{p,q,\epsilon}(n, k)$ is then defined to be the free k -module with basis

$$\mathcal{B} = \{Z_i^j \mid 1 \leq i, j \leq n\}.$$

We ordered \mathcal{B} by putting $Z_i^j > Z_u^v$ if either $i > u$, or $i = u$ and $j > v$. Define η by $\eta(Z_j^i) = j 3^{i-1}$. Besides, we put $[\cdot, \cdot] = 0$, and if $Z_i^l > Z_u^v$,

$$\langle Z_i^l, Z_u^v \rangle = \begin{cases} \epsilon_{vl}(p - q^{-1})Z_i^v \otimes Z_u^l, & \text{if } i > u \text{ and } l > v \\ 0, & \text{otherwise.} \end{cases} \quad (3.5)$$

$$S(Z_i^l \otimes Z_u^v) = \begin{cases} \epsilon_{vl}pZ_u^v \otimes Z_i^l, & \text{if } i = u, l > v, \\ \epsilon_{ui}qZ_u^v \otimes Z_i^l, & \text{if } l = v, i > u, \\ \epsilon_{iu}\epsilon_{vl}p^{-1}qZ_u^v \otimes Z_i^l, & \text{if } i > u, v > l \\ \epsilon_{vl}\epsilon_{ui}Z_u^v \otimes Z_i^l, & \text{if } i > u, l > v \end{cases}$$

To prove that $L_{p,q,\epsilon}(n, k)$ is a basic T -Lie algebra, since (3.5) it suffices to check the stability condition (2b) for $Z_i^l > Z_u^v$, such that $i > u$ and $l > v$:

$$(i - u)3^{l-1} > (i - u)3^{v-1}$$

then

$$i3^{l-1} + u3^{v-1} > i3^{v-1} + u3^{l-1}$$

but this equation has left side $\eta(Z_i^l) + \eta(Z_u^v)$ whereas the right side is $\eta(Z_i^v) + \eta(Z_u^l)$. This proves stability conditions.

IV. UNIVERSAL ENVELOPING ALGEBRAS

A. Construction of $U(L)$.

Definition IV.1. Let L be a T -Lie algebra with basis \mathcal{B} , and $\otimes_k L$ the k -tensor algebra of the module L . The universal enveloping algebra $U(L)$ is the quotient

$$U(L) = \otimes_k L / J$$

where J is the two sided ideal generated by

$$x \otimes y - T(x \otimes y) - [x, y], \quad x, y \in \mathcal{B}$$

Because the stability axiom (2b), the algebra $U(L)$ have a similar structure to a quadratic algebra with an ordering alghorithm [14] .

B. Examples.

Example IV.2.

$$U(L_{p,q,\epsilon}(n, k)) = M_{p,q,\epsilon}(n, k)$$

is a non-standard quantum deformation [12] of $GL(n)$.

Example IV.3.

$$U(sl_{n+1}^\pm)_q = U_q^\pm(sl_{n+1})$$

positive (negative) part of the Drinfeld-Jimbo quantum group of type A_n , $n = 3, 4$.

Example IV.4. In $U(\widetilde{sl_4^+})_q$ the equation $x_2x_6 = 0$ holds. Then $U(\widetilde{sl_4^+})_q$ is a enveloping algebra where the PBW theorem does not hold. So, if we want a good enveloping algebra we have to add conditions to the T -Lie algebras.

Besides, if β, \tilde{S} denotes the bracket and the symmetry of $(sl_4^+)_q$ respectively and the characteristic of the field \mathbb{K} is zero, then for $\gamma = Id - \tilde{S}$, where Id is the identity morphism on $(\widetilde{sl_4^+})_q^{2\otimes}$, the condition $\gamma(t) = 0$ implies $\beta(t) = 0$. Moreover, if \mathcal{B} is the canonical basis of $(\widetilde{sl_4^+})_q$, and x, y, z are arbitrary elements in \mathcal{B} , straightforward calculations (using *Mathematica* [15]) gives:

$$\begin{aligned} \beta(\beta(x, y), z) &= \beta(x, \beta(y, z)) - q_{x,y}\beta(y, \beta(x, z)) \\ \beta(z, \beta(x, y)) &= \beta(\beta(z, x), y) - q_{x,y}\beta(\beta(z, y), x) \end{aligned}$$

where $\tilde{S}(x \otimes y) = q_{x,y}y \otimes x$. This means that $(\widetilde{sl_4^+})_q$ has a structure of *balanced generalised Lie algebra* [9] and its universal enveloping algebra as such generalised Lie algebra is the same as T -Lie algebra. Therefore, the generalised Lie algebra axioms of Lyubashenko-Sudbery are not enough in order to obtain a PBW theorem.

Example IV.5. Let L be a basic T -Lie algebra. We are going to define a new T -Lie algebra L^0 : $L^0 = L$ in its structure of k -module, $[\cdot] = 0$, $\langle \cdot, \cdot \rangle = 0$, $\eta^0 = \eta$, $S^0 = S$ and define

$$\mathcal{S}(L) = U(L^0)$$

$\mathcal{S}(L)$ is a free k -module with basis the monomials formed by the products $z_{i_1}z_{i_2}\dots z_{i_r}$ of elements of \mathcal{B} such that $r \geq 0, z_{i_1} \leq z_{i_2} \leq \dots \leq z_{i_r}$ where $z_{i_j} = z_{i_{j+1}}$ if $q_{z_{i_j}z_{i_{j+1}}} = 1$.

Such an object $\mathcal{S}(L)$ will be called the *q-symmetric algebra of L*.

V. THE RELATIONSHIP BETWEEN UNIVERSAL ENVELOPING ALGEBRAS AND SYMMETRIC ALGEBRAS

Let L be a T -Lie algebra with basis \mathcal{B} , $x_\lambda \in \mathcal{B}$, $\Sigma = (x_{\lambda_1}, \dots, x_{\lambda_u})$ finite non-decreasing sequence of elements of \mathcal{B} . We write $z_\lambda = x_\lambda \in \mathcal{S}(L)$, $z_\Sigma = z_{\lambda_1} \dots z_{\lambda_u} \in \mathcal{S}(L)$, $z_\emptyset = 1 \in \mathcal{S}(L)$, $\eta(\lambda) = \eta(x_\lambda)$, $\eta(\Sigma) = \eta(z_\Sigma) = \eta(x_{\lambda_1}) + \dots + \eta(x_{\lambda_u})$. Besides we put $x_\lambda \leq \Sigma$ if $x_\lambda \leq x_{\lambda_1}$.

Lemma V.1 (A-B). Let L be a T -Lie algebra with basis \mathcal{B} . $\mathcal{P} = \mathcal{S}(L)$ q -symmetric algebra, \mathcal{P}_p the k -submodule generated by z_Σ such that $\eta(\Sigma) \leq p$. There is a k -morphism

$$_\cdot_\cdot : L \otimes_k \mathcal{P} \rightarrow \mathcal{P} \tag{5.6}$$

satisfying

- (A) $x_\lambda \cdot z_\Sigma = z_\lambda z_\Sigma$ for $x_\lambda \leq \Sigma$;
- (B) $x_\lambda \cdot z_\Sigma - z_\lambda z_\Sigma \in \mathcal{P}_{\eta(\lambda)+\eta(\Sigma)-1}$.

Proof. By induction on $\eta(\lambda) + \eta(\Sigma)$. If $\eta(\lambda) + \eta(\Sigma) = 1$ then $\eta(\lambda) = 1$ and $\Sigma = \emptyset$, it follows $z_\emptyset = 1$. Then define

$$x_\lambda \cdot 1 = z_\lambda$$

so (A) and (B) holds. Assume the existence of $x_{\lambda'} \cdot z_{\Sigma'}$ for $\eta(\lambda') + \eta(\Sigma') < \eta(\lambda) + \eta(\Sigma)$ satisfying (A) and (B). We have to define $x_\lambda \cdot z_\Sigma$.

There are two cases: $\lambda \leq \Sigma$ or $\lambda \not\leq \Sigma$.

Case $\lambda \leq \Sigma$: Because (A):

$$x_\lambda \cdot z_\Sigma = z_\lambda z_\Sigma$$

Case $\lambda \not\leq \Sigma$: We may write $\Sigma = (x_\mu, N)$ with $x_\mu \leq N$ and $x_\lambda > x_\mu$. Since $\eta(N) < \eta(\Sigma)$ and because at the induction hypothesis $x_\lambda \cdot z_N$ is already defined, and

$$w = x_\lambda \cdot z_N - z_\lambda z_N \in \mathcal{P}_{\eta(\lambda)+\eta(N)-1}.$$

Moreover, from $\eta(\mu) + \eta(\lambda) + \eta(N) - 1 < \eta(\mu) + \eta(\lambda) + \eta(N) = \eta(\lambda) + \eta(\Sigma)$ it follows that $x_\mu \cdot w$ is already defined.

We have

$$T(x_\lambda \otimes x_\mu) = q_{\lambda\mu} x_\mu \otimes x_\lambda + \sum_i \xi_i x_{\mu_i} \otimes x_{\lambda_i} \quad (5.7)$$

and because at (B) and the induction hypothesis $x_{\lambda_i} \cdot z_N \in \mathcal{P}_{\eta(\lambda_i)+\eta(N)}$. As a consequence $x_{\mu_i} \cdot (x_{\lambda_i} \cdot z_N)$ is already defined because $\eta(\mu_i) + \eta(\lambda_i) + \eta(N) < \eta(\lambda) + \eta(\mu) + \eta(N)$ according to stability axiom.

We may define

$$x_\lambda \cdot z_\Sigma = q_{\lambda\mu} z_\mu z_\lambda z_N + q_{\lambda\mu} x_\mu \cdot w + \sum_i \xi_i x_{\mu_i} \cdot (x_{\lambda_i} \cdot z_N) + [x_\lambda, x_\mu] \cdot z_N \quad (5.8)$$

where $w = x_\lambda \cdot z_N - z_\lambda z_N$; $[x_\lambda, x_\mu] \cdot z_N$ is already defined because $\eta([x_\lambda, x_\mu]) + \eta(N) < \eta(\lambda) + \eta(\mu) + \eta(N) = \eta(\lambda) + \eta(\Sigma)$.

Now only remains to prove (B). From $z_\lambda z_\Sigma = q_{\lambda\mu} z_\mu z_\lambda z_N$ we obtain

$$x_\lambda \cdot z_\Sigma - z_\lambda z_\Sigma = q_{\lambda\mu} x_\mu \cdot w + \sum_i \xi_i x_{\mu_i} \cdot x_{\lambda_i} \cdot z_N + [x_\lambda, x_\mu] \cdot z_N.$$

Moreover

$$x_\mu \cdot w \in \mathcal{P}_{\eta(\mu)+\eta(w)} = \mathcal{P}_{\eta(\mu)+\eta(\lambda)+\eta(N)-1} = \mathcal{P}_{\eta(\lambda)+\eta(\Sigma)-1}$$

$$x_{\mu_i} \cdot x_{\lambda_i} \cdot z_N \in \mathcal{P}_{\eta(\mu_i)+\eta(\lambda_i)+\eta(N)} \subseteq \mathcal{P}_{\eta(\mu)+\eta(\lambda)-1+\eta(N)} = \mathcal{P}_{\eta(\lambda)+\eta(\Sigma)-1}$$

$$[x_\lambda, x_\mu] \cdot z_N \in \mathcal{P}_{\eta(\lambda)+\eta(\mu)-1+\eta(N)} = \mathcal{P}_{\eta(\lambda)+\eta(\Sigma)-1}$$

imply

$$x_\lambda \cdot z_\Sigma - z_\lambda z_\Sigma \in \mathcal{P}_{\eta(\lambda)+\eta(\Sigma)-1}$$

□

Definition V.1. Let L be a T -Lie algebra with basis \mathcal{B} . We call L adequate if the morphism from the lemma (A-B) is such that the condition

$$x_{\lambda'} \cdot x_{\mu'} \cdot z_M - T(x_{\lambda'} \otimes x_{\mu'}) \cdot z_M = [x_{\lambda'}, x_{\mu'}] \cdot z_M \quad (5.9)$$

for all $\eta(x_{\lambda'}) + \eta(x_{\mu'}) + \eta(M) \leq r$ implies

$$\begin{aligned} & \langle x_{\lambda}, x_{\mu} \rangle \cdot x_{\gamma} \cdot z_N - q_{\lambda\mu} [x_{\mu}, [x_{\lambda}, x_{\gamma}]] \cdot z_N = \\ & q_{\mu\gamma} q_{\lambda\gamma} \langle x_{\gamma} [x_{\lambda}, x_{\mu}] \rangle \cdot z_N + q_{\mu\gamma} q_{\lambda\gamma} x_{\gamma} \cdot \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N \\ & + q_{\mu\gamma} \langle x_{\lambda}, x_{\gamma} \rangle \cdot x_{\mu} \cdot z_N - q_{\lambda\mu} x_{\mu} \cdot \langle x_{\lambda}, x_{\gamma} \rangle \cdot z_N \\ & + q_{\mu\gamma} [x_{\lambda}, x_{\gamma}] \cdot x_{\mu} \cdot z_N - q_{\lambda\mu} x_{\mu} \cdot [x_{\lambda}, x_{\gamma}] \cdot z_N \\ & + x_{\lambda} \cdot \langle x_{\mu}, x_{\gamma} \rangle \cdot z_N - q_{\lambda\mu} q_{\lambda\gamma} \langle x_{\mu}, x_{\gamma} \rangle \cdot x_{\lambda} \cdot z_N - q_{\lambda\gamma} q_{\lambda\mu} \langle [x_{\mu}, x_{\gamma}], x_{\lambda} \rangle \cdot z_N \end{aligned} \quad (5.10)$$

for every $x_{\lambda} > x_{\mu} > x_{\gamma} \in \mathcal{B}$, $x_{\gamma} \leq z_N$ such that $\eta(x_{\lambda}) + \eta(x_{\mu}) + \eta(x_{\gamma}) + \eta(N) \leq r+1$.

Lemma V.2 (C). Let L be an adequate T -Lie algebra with basis \mathcal{B} , and \mathcal{P} the related q -symmetric algebra. Then there exists a k -morphism $_ \cdot _ : L \otimes_k \mathcal{P} \rightarrow \mathcal{P}$ such that

$$(C) \quad x_{\lambda} \cdot x_{\mu} \cdot z_N = T(x_{\lambda} \otimes x_{\mu}) \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N, \quad \forall z_N \in \mathcal{P}, \forall x_{\lambda}, x_{\mu} \in \mathcal{B}.$$

Proof. Let $_ \cdot _$ be the morphism from lemma (A-B). There are two cases:

1. $\mu \leq N$ or $\lambda \leq N$;
2. $\mu \not\leq N$ and $\lambda \not\leq N$;

(1): Assume $\mu \leq N$ and $\mu < \lambda$. Let $M = (\mu, N)$, then, by definition

$$\begin{aligned} x_{\lambda} \cdot x_{\mu} \cdot z_N &= x_{\lambda} z_M \text{ where } \lambda \not\leq M \\ &= z_{\lambda} \cdot z_M + q_{\lambda\mu} x_{\mu} \cdot w + \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N \end{aligned}$$

On the other hand,

$$\begin{aligned} & T(x_{\lambda} \otimes x_{\mu}) \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N \\ &= q_{\lambda\mu} x_{\mu} \cdot x_{\lambda} \cdot z_N + \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N \\ &= q_{\lambda\mu} x_{\mu} \cdot (z_{\lambda} z_N) + q_{\lambda\mu} x_{\mu} \cdot w + \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N \end{aligned}$$

since $\mu < \lambda$ and $\mu \leq N$ it holds $z_{\lambda} z_N = c z_{N'}$ where $\mu \leq N'$ and $c \in k$,

$$x_{\mu} \cdot (z_{\lambda} z_N) = c x_{\mu} \cdot z_{N'} = c z_{\mu} z_{N'} = z_{\mu} z_{\lambda} z_N,$$

so

$$q_{\lambda\mu} x_{\mu} \cdot (z_{\lambda} z_N) = q_{\lambda\mu} z_{\mu} z_{\lambda} z_N = z_{\lambda} z_{\mu} z_N = z_{\lambda} z_M,$$

therefore

$$\begin{aligned} & T(x_{\lambda} \otimes x_{\mu}) \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N \\ &= z_{\lambda} z_M + q_{\lambda\mu} x_{\mu} \cdot w + \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N + [x_{\lambda}, x_{\mu}] \cdot z_N \\ &= x_{\lambda} \cdot x_{\mu} \cdot z_N \end{aligned}$$

(i.e. (C) holds for $\mu < \lambda$). It follows, multiplying by $-q_{\mu\lambda}$:

$$-q_{\mu\lambda} x_{\lambda} \cdot x_{\mu} \cdot z_N = -x_{\mu} \otimes x_{\lambda} \cdot z_N - q_{\mu\lambda} \langle x_{\lambda}, x_{\mu} \rangle \cdot z_N - q_{\mu\lambda} [x_{\lambda}, x_{\mu}] \cdot z_N. \quad (5.11)$$

This implies, using antisymmetry,

$$x_{\mu} \cdot x_{\lambda} \cdot z_N - T(x_{\mu} \otimes x_{\lambda}) \cdot z_N = [x_{\mu}, x_{\lambda}] \cdot z_N \quad (5.12)$$

and we conclude that (C) also holds for $\lambda < \mu$.

(2): Let $N = (\gamma, Q)$ where $\gamma \leq Q$, $\gamma < \lambda$, $\gamma < \mu$. We proceed by induction on $\eta(\lambda) + \eta(\mu) + \eta(N)$. Suppose that for each $\eta(\lambda') + \eta(\mu') + \eta(N') \leq r$ it holds (C). Then, for $\eta(\lambda) + \eta(\mu) + \eta(N) \leq r + 1$ we have:

$$x_\mu \cdot z_N = x_\mu \cdot (x_\gamma \cdot z_Q) = T(x_\mu \otimes x_\gamma) \cdot z_Q + [x_\mu, x_\gamma] \cdot z_Q \quad (5.13)$$

because $\eta(\mu) + \eta(\gamma) + \eta(Q) = \eta(\mu) + \eta(N) \leq r$ and the induction hypothesis.

Now, $x_\mu \cdot z_Q = z_\mu z_Q + w$ where $w \in \mathcal{P}_{\eta(\mu)+\eta(Q)-1}$. We may apply (C) to $x_\lambda \cdot x_\gamma \cdot (z_\mu z_Q)$ since $z_\mu z_Q = cz_{Q'}$ where $c \in k$ and $\gamma \leq Q'$ because $\gamma \leq Q$, $\gamma < \mu$ and case (A).

Also (C) applies to $x_\lambda \cdot x_\gamma \cdot w$ since

$$\begin{aligned} \eta(\lambda) + \eta(\gamma) + \eta(w) &\leq \eta(\lambda) + \eta(\gamma) + \eta(\mu) + \eta(Q) - 1 \\ &= \eta(\lambda) + \eta(\gamma) + \eta(N) - 1 \leq r \end{aligned}$$

and the induction hypothesis.

The preceding remarks show that (C) applies to

$$x_\lambda \cdot x_\gamma \cdot x_\mu \cdot z_Q = x_\lambda \cdot x_\gamma \cdot (z_\mu z_Q) + x_\lambda \cdot x_\gamma \cdot w$$

Using (5.13) and multiplying by x_λ ,

$$\begin{aligned} x_\lambda \cdot x_\mu \cdot z_N &= x_\lambda \cdot T(x_\mu \otimes x_\gamma) \cdot z_Q + x_\lambda \cdot [x_\mu, x_\gamma] \cdot z_Q \\ &= \underbrace{q_{\mu\gamma} x_\lambda \cdot x_\gamma \cdot x_\mu \cdot z_Q}_{= q_{\mu\gamma} q_{\lambda\gamma} x_\gamma \cdot x_\lambda \cdot x_\mu \cdot z_Q} + x_\lambda \cdot \langle x_\mu, x_\gamma \rangle \cdot z_Q + x_\lambda \cdot [x_\mu, x_\gamma] \cdot z_Q \\ &= q_{\mu\gamma} q_{\lambda\gamma} x_\gamma \cdot x_\lambda \cdot x_\mu \cdot z_Q + q_{\mu\gamma} \langle x_\lambda, x_\gamma \rangle \cdot x_\mu \cdot z_Q \\ &\quad + q_{\mu\gamma} [x_\lambda, x_\gamma] \cdot x_\mu \cdot z_Q + x_\lambda \cdot \langle x_\mu, x_\gamma \rangle \cdot z_Q + x_\lambda \cdot [x_\mu, x_\gamma] \cdot z_Q. \end{aligned}$$

Recall that λ, μ are interchangeable:

$$\begin{aligned} x_\mu \cdot x_\lambda \cdot z_N &= q_{\lambda\gamma} q_{\mu\gamma} x_\gamma \cdot x_\mu \cdot x_\lambda \cdot z_Q + q_{\lambda\gamma} \langle x_\mu, x_\gamma \rangle \cdot x_\lambda \cdot z_Q \\ &\quad + q_{\lambda\gamma} [x_\mu, x_\gamma] \cdot x_\lambda \cdot z_Q + x_\mu \cdot \langle x_\lambda, x_\gamma \rangle \cdot z_Q + x_\mu \cdot [x_\lambda, x_\gamma] \cdot z_Q. \end{aligned}$$

Now use $\eta(\lambda) + \eta(\mu) + \eta(Q) = \eta(\lambda) + \eta(N) \leq r$ to obtain

$$\begin{aligned} x_\lambda \cdot x_\mu \cdot z_N - q_{\lambda\mu} x_\mu \cdot x_\lambda \cdot z_N &= \\ &\quad q_{\mu\gamma} q_{\lambda\gamma} x_\gamma \cdot ([x_\lambda, x_\mu] + \langle x_\lambda, x_\mu \rangle) \cdot z_Q + \\ &\quad q_{\mu\gamma} \langle x_\lambda, x_\gamma \rangle \cdot x_\mu \cdot z_Q - q_{\lambda\mu} x_\mu \cdot \langle x_\lambda, x_\gamma \rangle \cdot z_Q + \\ &\quad q_{\mu\gamma} [x_\lambda, x_\gamma] \cdot x_\mu \cdot z_Q - q_{\lambda\mu} x_\mu \cdot [x_\lambda, x_\gamma] \cdot z_Q + \\ &\quad x_\lambda \cdot \langle x_\mu, x_\gamma \rangle \cdot z_Q - q_{\lambda\mu} q_{\lambda\gamma} \langle x_\mu, x_\gamma \rangle \cdot x_\lambda \cdot z_Q + \\ &\quad x_\lambda \cdot [x_\mu, x_\gamma] \cdot z_Q - q_{\lambda\mu} q_{\lambda\gamma} [x_\mu, x_\gamma] \cdot x_\lambda \cdot z_Q. \end{aligned} \quad (5.14)$$

Furthermore,

$$\begin{aligned} x_\lambda \cdot [x_\mu, x_\gamma] \cdot z_Q - q_{\lambda\mu} q_{\lambda\gamma} [x_\mu, x_\gamma] \cdot x_\lambda \cdot z_Q &= \\ &= -q_{\mu\gamma} x_\lambda \cdot [x_\gamma, x_\mu] + q_{\lambda\mu} q_{\lambda\gamma} q_{\mu\gamma} [x_\gamma, x_\mu] \cdot x_\lambda \cdot z_Q \\ &= q_{\mu\gamma} q_{\lambda\gamma} q_{\lambda\mu} ([x_\gamma, x_\mu] \cdot x_\lambda \cdot z_Q - q_{\gamma\lambda} q_{\mu\lambda} x_\lambda \cdot [x_\gamma, x_\mu] \cdot z_Q). \end{aligned} \quad (5.15)$$

If we suppose $x_\mu < x_\lambda$ then we can make use of multiplicativity condition and since $\eta([x_\gamma, x_\mu]) + \eta(\lambda) + \eta(Q) < \eta(\lambda) + \eta(\mu) + \eta(\gamma) + \eta(Q) = \eta(\lambda) + \eta(\mu) + \eta(N)$ we obtain that (5.15) is equal to

$$\begin{aligned} &= q_{\mu\gamma}q_{\lambda\gamma}q_{\lambda\mu}[[x_\gamma, x_\mu], x_\lambda] \cdot z_Q + q_{\mu\gamma}q_{\lambda\gamma}q_{\lambda\mu}\langle [x_\gamma, x_\mu], x_\lambda \rangle \cdot z_Q \\ &= -q_{\lambda\gamma}q_{\lambda\mu}[[x_\mu, x_\gamma], x_\lambda] \cdot z_Q - q_{\lambda\gamma}q_{\lambda\mu}\langle [x_\mu, x_\gamma], x_\lambda \rangle \cdot z_Q. \end{aligned} \quad (5.16)$$

Using multiplicativity again and since $\eta(x_\gamma) + \eta([x_\lambda, x_\mu]) + \eta(Q) < \eta(x_\lambda) + \eta(\mu) + \eta(\gamma) + \eta(Q) = \eta(x_\lambda) + \eta(x_\mu) + \eta(N)$ we may write

$$\begin{aligned} x_\gamma \cdot [x_\lambda, x_\mu] \cdot z_Q &= \\ q_{\gamma\mu}q_{\gamma\lambda}[x_\lambda, x_\mu] \cdot x_\gamma \cdot z_Q &+ [x_\gamma, [x_\lambda, x_\mu]] \cdot z_Q + \langle x_\gamma, [x_\lambda, x_\mu] \rangle \cdot z_Q \end{aligned} \quad (5.17)$$

Sustitute (5.16) and (5.17) in (5.14),

$$\begin{aligned} x_\lambda \cdot x_\mu \cdot z_N - q_{\mu\lambda}x_\mu \cdot x_\lambda \cdot z_N &= \\ [x_\lambda, x_\mu] \cdot x_\gamma \cdot z_Q &+ q_{\mu\gamma}q_{\lambda\gamma}[x_\gamma, [x_\lambda, x_\mu]] \cdot z_Q \\ &+ q_{\mu\gamma}q_{\lambda\gamma}\langle x_\gamma, [x_\lambda, x_\mu] \rangle \cdot z_Q + q_{\mu\gamma}q_{\lambda\gamma}x_\gamma \cdot \langle x_\lambda, x_\mu \rangle \cdot z_Q \\ &+ q_{\mu\gamma}\langle x_\lambda, x_\gamma \rangle \cdot x_\mu \cdot z_Q - q_{\lambda\mu}x_\mu \langle x_\lambda, x_\gamma \rangle \cdot z_Q \\ &+ q_{\mu\gamma}[x_\lambda, x_\gamma] \cdot x_\mu \cdot z_Q - q_{\lambda\mu}x_\mu[x_\lambda, x_\gamma] \cdot z_Q \\ &+ x_\lambda \cdot \langle x_\mu, x_\gamma \rangle \cdot z_Q - q_{\lambda\mu}q_{\lambda\gamma}\langle x_\mu, x_\gamma \rangle \cdot x_\lambda \cdot z_Q \\ &- q_{\lambda\gamma}q_{\lambda\mu}[[x_\mu, x_\gamma], x_\lambda] \cdot z_Q - q_{\lambda\gamma}q_{\lambda\mu}\langle [x_\mu, x_\gamma], x_\lambda \rangle \cdot z_Q \end{aligned}$$

since L is adequate,

$$\begin{aligned} &= [x_\lambda, x_\mu] \cdot x_\gamma \cdot z_Q + \langle x_\lambda, x_\mu \rangle \cdot x_\gamma \cdot z_Q + \\ &q_{\mu\gamma}q_{\lambda\gamma}[x_\gamma, [x_\lambda, x_\mu]] \cdot z_Q - q_{\lambda\gamma}q_{\lambda\mu}[[x_\mu, x_\gamma], x_\lambda] \cdot z_Q - q_{\lambda\mu}[x_\mu, [x_\lambda, x_\gamma]] \cdot z_Q. \end{aligned}$$

Thanks to Jacobi identity and (A) we get

$$x_\lambda \cdot x_\mu \cdot z_N - q_{\lambda\mu}x_\mu \cdot x_\lambda \cdot z_N - \langle x_\lambda, x_\mu \rangle \cdot z_N = [x_\lambda, x_\mu] \cdot z_N \quad (5.18)$$

if $x_\mu < x_\lambda$.

Multiplying both sides of (5.18) by $-q_{\lambda\mu}$ and using antisymmetry, we get

$$x_\mu \cdot x_\lambda \cdot z_N - q_{\lambda\mu}x_\lambda \cdot x_\mu \cdot z_N - \langle x_\mu, x_\lambda \rangle \cdot z_N = [x_\mu, x_\lambda] \cdot z_N$$

so (5.18) also holds if $x_\lambda < x_\mu$. \square

VI. REPRESENTATIONS

Definition VI.1. Let L be a basic T -Lie algebra and V a k -module. A k -morphism $_\cdot_\cdot : L \otimes_k V \rightarrow V$ is called representation of L if it satisfies

$$x \cdot y \cdot v - T(x \otimes y) \cdot v = [x, y] \cdot v, \quad \forall x, y \in L, \forall v \in V$$

where $(a \otimes b) \cdot v$ means $a \cdot b \cdot v$.

Theorem VI.1. If L is an adequate basic T -Lie algebra then L has a natural representation on its q -symmetric algebra $\mathcal{S}(L)$.

Corollary VI.2. If L is an adequate basic T -Lie algebra then its universal enveloping algebra $U(L)$ has a representation on the q -symmetric algebra $\mathcal{S}(L)$.

Inside $(sl_n^+)_q$, $n \geq 4$, the k -submodules generated by the basic elements given at Figure 5 have a structure of basic T -Lie algebra that looks like $(sl_4^+)_q$. But such

$$\begin{array}{ccccc} & e_{ij} & & e_{jk} & & e_{kl} \\ & & e_{ik} & & e_{jl} & \\ & & & e_{il} & & \end{array}$$

FIGURE 5. A basic T -Lie algebra of type $(sl_4^+)_q$

algebra have a graduation given by $\eta(e_{ab}) = a(b - a)$, this in not, in general, the graduation of $(sl_4^+)_q$. However, Figure 5 still is a basic T -Lie algebra.

The basic T -Lie algebras given by Figure 5 will be called *of type $(sl_4^+)_q$* . In a similar way we may define *basic T -Lie algebras of type $(sl_n^\pm)_q$* .

Example VI.3. Every basic T -Lie algebra of type $(sl_4^\pm)_q$ is adequate.

Proof. Suppose $x_\lambda > x_\mu > x_\gamma \in \mathcal{B}$, $x_\gamma \leq z_N$ such that $\eta(x_\lambda) + \eta(x_\mu) + \eta(x_\gamma) + \eta(N) \leq r + 1$. We have to prove that (5.10) holds.

Note that $\langle x_\lambda, x_\mu \rangle = 0$ for any $x_\lambda, x_\mu \in \mathcal{B}$ except e_{jk}, e_{il} , so each term in the equation (5.10) vanishes or e_{jk}, e_{il} appears. This means that the equation (5.10) holds trivially except in the following cases:

$$e_{ij} < e_{jk} < e_{jl}, e_{ij} < e_{ik} < e_{jl}, e_{ik} < e_{jk} < e_{kl}, e_{ik} < e_{jl} < e_{kl}$$

Case $e_{ij} < e_{jk} < e_{jl}$: The left side of (5.10) vanishes whereas the right side is:

$$\begin{aligned} e_{jk} \cdot e_{il} \cdot z_N - q^2 e_{il} \cdot e_{jk} \cdot z_N + q \langle e_{ik}, e_{jl} \rangle \cdot z_N \\ = e_{jk} \cdot e_{il} \cdot z_N - q^2 e_{il} \cdot e_{jk} \cdot z_N + (q^2 - 1) e_{il} \cdot e_{jk} \cdot z_N = 0, \end{aligned}$$

because $e_{il} \cdot e_{jk} \cdot z_N = e_{jk} \cdot e_{il} \cdot z_N$ since $\eta(e_{il}) + \eta(e_{jk}) < \eta(e_{ij}) + \eta(e_{jk}) + \eta(e_{jl})$ and supposition (5.9).

Case $e_{ij} < e_{ik} < e_{jl}$: Let be $d = \eta(e_{ij}) + \eta(e_{ik}) + \eta(e_{jl})$. The left side of (5.10) is

$$\begin{aligned} (q^{-1} - q) e_{il} \cdot e_{jk} \cdot e_{ij} \cdot z_N \\ = (q^{-1} - q) (q e_{il} \cdot e_{ij} \cdot e_{jk} \cdot z_N - q e_{il} \cdot e_{ik} \cdot z_N) \\ = (q^{-1} - q) (e_{ij} \cdot e_{il} \cdot e_{jk} \cdot z_N) - (q^{-1} - q) q e_{il} \cdot e_{ik} \cdot z_N \\ (\eta(e_{il}) + \eta(e_{ij}) + \eta(e_{jk}) < d \text{ and (5.9)}) \\ = (q^{-1} - q) (e_{ij} \cdot e_{il} \cdot e_{jk} \cdot z_N - e_{il} \cdot e_{ik} \cdot z_N + q e_{ik} \cdot e_{il} \cdot z_N) \end{aligned}$$

($\eta(e_{ik}) + \eta(e_{il}) < d$), and this is the right side of (5.10).

The remainig cases are similar. \square

Example VI.4. Every basic T -Lie algebra of type $(sl_n^\pm)_q$ is adequate, $n = 5, 6$.

Proof. By similar calculations as in the previous example. \square

Note that the symbol $\cdot z_N$ is redundant in calculations at example VI.3. This remark leads to the following lemma.

Let $\otimes_k L$ be the tensorial k -algebra and J_r the k -submodule generated by

$$x_\alpha \otimes x_\beta \otimes x_\delta - T(x_\alpha \otimes x_\beta) \otimes x_\delta - [x_\alpha, x_\beta] \otimes x_\delta, \quad (6.19)$$

$$x_\alpha \otimes x_\beta \otimes x_\delta - x_\alpha \otimes T(x_\beta \otimes x_\delta) - x_\alpha \otimes [x_\beta, x_\delta] \quad (6.20)$$

for $\eta(\alpha) + \eta(\beta) + \eta(\delta) \leq r$, $\forall x_\alpha, x_\beta, x_\delta \in \mathcal{B}$.

Lemma VI.2. L is adequate if

$$\begin{aligned} \langle x_\lambda, x_\mu \rangle \otimes x_\gamma - q_{\lambda\mu}[x_\mu, [x_\lambda, x_\gamma]] \equiv & \\ & q_{\mu\gamma}q_{\lambda\gamma}\langle x_\gamma[x_\lambda, x_\mu] \rangle + q_{\mu\gamma}q_{\lambda\gamma}x_\gamma \otimes \langle x_\lambda, x_\mu \rangle \\ & + q_{\mu\gamma}\langle x_\lambda, x_\gamma \rangle \otimes x_\mu - q_{\lambda\mu}x_\mu \otimes \langle x_\lambda, x_\gamma \rangle \\ & + q_{\mu\gamma}[x_\lambda, x_\gamma] \otimes x_\mu - q_{\lambda\mu}x_\mu \otimes [x_\lambda, x_\gamma] \\ & + x_\lambda \otimes \langle x_\mu, x_\gamma \rangle - q_{\lambda\mu}q_{\lambda\gamma}\langle x_\mu, x_\gamma \rangle \otimes x_\lambda - q_{\lambda\gamma}q_{\lambda\mu}\langle [x_\mu, x_\gamma], x_\lambda \rangle \pmod{J_r} \end{aligned} \quad (6.21)$$

for every $x_\lambda > x_\mu > x_\gamma \in \mathcal{B}$ such that $\eta(\lambda) + \eta(\mu) + \eta(\gamma) \leq r + 1$.

VII. POINCARÉ-BIRKHOFF-WITT THEOREM

Let us define

$$T^n = \underbrace{L \otimes_k \dots \otimes_k L}_{n\text{-times}}$$

For $u = x_{\lambda_1} \otimes \dots \otimes x_{\lambda_n} \in T^n$ define $\delta(u) = \eta(x_{\lambda_1}) + \dots + \eta(x_{\lambda_n})$, $D(u) = \#\{(x_{\lambda_i}, x_{\lambda_j}) \mid x_{\lambda_i} > x_{\lambda_j} \text{ and } i < j\}$. If $u \in \otimes_k L$ and $u = \sum_i \xi_i u_i$ with $u_i \in T^i$, $\xi_i \in k$, $\forall i$, let us put

$$D(u) = \max\{D(u_i) \mid \xi_i \neq 0, i\} \quad (7.22)$$

$$\delta(u) = \max\{\delta(u_i) \mid \xi_i \neq 0, i\} \quad (7.23)$$

The number $D(u)$ is called *the disorder of u* .

Denote by \mathcal{T}_p the k -submodule generated by $u \in \otimes_k L$ such that $\delta(u) \leq p$.

Definition VII.1. A sequence $(x_{\lambda_1}, \dots, x_{\lambda_n})$ of elements in a basis of a basic T -Lie algebra is called *non-decreasing* if $x_{\lambda_1} \leq \dots \leq x_{\lambda_n}$ and $x_{\lambda_i} = x_{\lambda_{i+1}}$ if and only if $S(x_{\lambda_i} \otimes x_{\lambda_{i+1}}) = x_{\lambda_i} \otimes x_{\lambda_{i+1}}$.

Theorem VII.1 (Poincaré-Birkhoff-Witt). *Let L be an adequate T -Lie algebra with a basis \mathcal{B} . The monomials formed by finite non-decreasing sequences of elements in \mathcal{B} constitute a free k -basis of the universal enveloping algebra $U(L)$.*

Proof. Let $P : \otimes_k L \rightarrow U(L)$ be the canonical k -morphism, \mathcal{M} the k -submodule generated by the monomials described in the formulation of the theorem. We have to prove that $U(L) = \mathcal{M}$. Note that

$$U(L) = \sum_{p=1}^{\infty} P(\mathcal{T}_p)$$

If $p = 1$ then $\mathcal{T}_p \subseteq L$ it follows $P(\mathcal{T}_p) \subseteq \mathcal{M}$. Suppose $P(\mathcal{T}_r) \subseteq \mathcal{M}$. It suffices to show that $P(\mathcal{T}_{r+1}) \subseteq \mathcal{M}$.

Define \mathcal{T}_r^u as the k -submodule of \mathcal{T}_r generated by elements with disorder $\leq u$, and proceed by a second induction on the disorder. We have $P(\mathcal{T}_{r+1}^0) \subseteq \mathcal{M}$. Suppose

$v = a \otimes x \otimes y \otimes b \in \mathcal{T}_{r+1}^u$ where $x > y \in \mathcal{B}$, and $a \in \mathcal{T}^n$, $b \in \mathcal{T}^m$ monomials form by basic elements in \mathcal{B} . Then

$$\begin{aligned} P(v) &= P(a \otimes q_{xy}y \otimes x \otimes b) + P(a \otimes \langle x, y \rangle \otimes b) + P(a \otimes [x, y] \otimes b) \\ &\equiv P(a \otimes q_{xy}y \otimes x \otimes b) \pmod{\mathcal{T}_r} \end{aligned}$$

but $P(a \otimes q_{xy}y \otimes x \otimes b) \in \mathcal{T}_{r+1}^{u-1} \subseteq \mathcal{M}$. Hence $P(v) \in \mathcal{M}$ it follows $P(\mathcal{T}_{r+1}) \subseteq \mathcal{M}$.

It remains to prove linear independence. For a given sequence $\Sigma = (x_{\lambda_1}, \dots, x_{\lambda_n})$ of non-decreasing elements of \mathcal{B} , define $x_\Sigma = x_{\lambda_1} \dots x_{\lambda_n} \in U(L)$. Suppose

$$\sum_i \xi_i x_{\Sigma_i} = 0$$

where each Σ_i is a sequence non-decreasing and $\xi_i \in k$, $\forall i$. Using the representation of $U(L)$, we get from lemma (C)

$$0 = \sum_i \xi_i x_{\Sigma_i} \cdot 1 = \sum_i \xi_i z_{\Sigma_i}$$

and because the linear independence of the $z_{\Sigma_i} \in \mathcal{S}(L)$, it follows that $\xi_i = 0$, $\forall i$. \square

Corollary VII.2. $U(sl_n^\pm)_q$ has a basis of the Poincaré-Birkhoff-Witt type, $n = 2, 3, 4, 5$.

VIII. BRAIDS

Proposition VIII.1. In $L = (sl_n^\pm)_q$, $n = 2, 3, 4$ it holds the braid equation:

$$T_{12}T_{23}T_{12}|_{^3L} = T_{23}T_{12}T_{23}|_{^3L},$$

where $T_{12} = T \otimes_k Id_L$, $T_{23} = Id_L \otimes_k T$.

Proof. By straightforward calculations on the basic elements. (Using *Mathematica* [15]). \square

Proposition VIII.2. The presymmetry S of a T -Lie algebras holds the braid equation

$$S_{12}S_{23}S_{12} = S_{23}S_{12}S_{23}$$

Proof. Let x, y, z be basic elements. Then

$$S_{12}S_{23}S_{12}(x \otimes y \otimes z) = q_{x,y}q_{x,z}q_{y,z}z \otimes y \otimes x = S_{23}S_{12}S_{23}(x \otimes y \otimes z)$$

\square

Remark VIII.1. The symmetry of $\widetilde{(sl_4^+)_q}$ is a braid morphism, however we have no PBW theorem for $U(\widetilde{sl_4^+}_q)$. As a consequence the PBW theorem is independent from the braid equation.

IX. NON-STANDARD QUANTUM DEFORMATIONS OF $GL(n)$

Definition IX.1. Let p, q be units in a commutative unitary ring k with $pq \neq 1$ and choose $\alpha(\alpha - 1)/2$ discrete parameters ϵ_{ij} , $\epsilon_{ij} = \pm 1$, $1 \leq i < j \leq \alpha$, $\epsilon_{ii} = 1$, $\epsilon_{ji} = \epsilon_{ij}$. Let m, n be positive integers such that $m, n \leq \alpha$

The k -module $L_{p,q,\epsilon}(n, m, k)$ is then defined to be the free k -module with basis

$$\mathcal{B} = \{Z_i^j \mid 1 \leq i \leq n, 1 \leq j \leq m\}.$$

Now define an order on \mathcal{B} and morphisms $S, T, \langle, \rangle, [,]$ copying the structure of $L(n, k)$ in example III.6.

Proposition IX.2. $L_{p,q,\epsilon}(n, m, k)$ has a structure of basic T -Lie algebra.

In a similar way to the algebras of type $(sl_n^+)_q$, (see section VI) we can define algebras of type $L_{p,q,\epsilon}(n, m, k)$.

Lemma IX.1. Every algebra of type $L_{p,q,\epsilon}(\lambda, \mu, k)$ is an adequate basic T -Lie algebra, where $\lambda, \mu \in \{2, 3\}$.

Lemma IX.2. If $Z_u^v > Z_i^j > Z_a^b$ then there exists L being a T -Lie subalgebra of $L_{p,q,\epsilon}(n, m, k)$ and numbers $\lambda, \mu \in \{2, 3\}$ such that L is of type $L_{p,q,\epsilon'}(\lambda, \mu, k)$ and $\{Z_u^v, Z_i^j, Z_a^b\} \subset L$.

Proof. Let us put the basic elements in a matrix array (Figure 6 (a)).

$$\begin{array}{cccc} & Z_1^1 & Z_1^2 & \dots & Z_1^n \\ \text{(a)} & \vdots & \vdots & & \vdots \\ & Z_m^1 & Z_m^2 & \dots & Z_m^n \end{array} \quad \text{(b)} \quad \begin{array}{cc} \circ_i^j & \circ_i^{j+u} \\ \circ_{i+u}^j & \circ_{i+u}^{j+u} \end{array}$$

FIGURE 6. (a) The basic T -Lie algebra $L_{p,q,\epsilon'}(\lambda, \mu, k)$. (b) Diagonal relationship.

Note that for positive integers u, v the elements appearing in the pseudobacket definition are in a diagonal relationship (Figure 6(b)), and they form a free basis of a T -Lie algebra of type $L_{p,q,\epsilon'}(2, 2, k)$, where $\epsilon' = \{1, \epsilon_{i,i+u}, \epsilon_{j,j+u}\}$.

For $Z_u^v > Z_i^j > Z_a^b$ there are several cases. The cases given by Figure 7(a), 7(b), 7(c), or they form a triangle which can be fitted, with vertices on the border, inside of the rectangle at Figure 7(d).

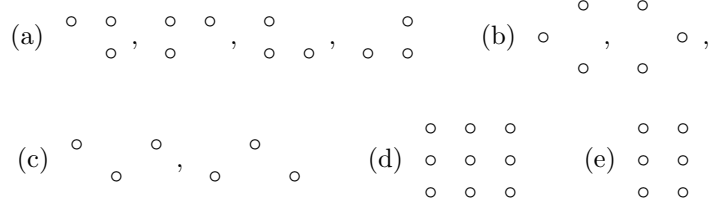
In the case given by Figure 7(a) we may complete each triangle to a square and obtain $L_{p,q,\epsilon_0}(2, 2, k)$. In the case given by Figure 7(b), each triangle can be completed to a rectangle in the form of Figure 7(e) and we get $L_{p,q,\epsilon_1}(3, 2, k)$. Similarly in the case given by Figure 7(c) we get $L_{p,q,\epsilon_2}(2, 3, k)$. Finally, in the case given by Figure 7(d), we obtain $L_{p,q,\epsilon_3}(3, 3, k)$. \square

Theorem IX.3. $L_{p,q,\epsilon}(n, m, k)$ is an adequate basic T -Lie algebra.

Corollary IX.3. The monomials formed by non-decreasing finite sequences of elements in

$$\mathcal{B} = \{Z_i^j \mid 1 \leq i \leq n, 1 \leq j \leq m\}$$

constitute a free basis of the k -module $M_{p,q,\epsilon}(n, m, k) = U L_{p,q,\epsilon}(n, m, k)$.

FIGURE 7. Some cases in $L_{p,q,\epsilon}(n, m, k)$ X. $U(sl_{n+1}^+)_q$. THE GENERAL CASE.

Lemma X.1. Let e_{ab}, e_{uv}, e_{ij} be basic elements in $(sl_{n+1}^+)_q$ and $[\cdot, \cdot]$ usual bracket in sl_{n+1} .

1. $e_{ab} < e_{uv}$ if and only if $a + b < i + j$ or $a + b = i + j$ and $b < j$.
2. If $S(e_{ab} \otimes e_{uv}) = q^{c_{ab,uv}} e_{uv} \otimes e_{ab}$ and $e_{ab} < e_{uv}$ then

$$c_{ab,uv} = -\delta_{v,a} + \delta_{v,b} + \delta_{u,a} - \delta_{u,b}$$

3. If $e_{ab} < e_{uv} < e_{ij}$ then

$$q^{c_{uv,ab}} [e_{uv}, [e_{ab}, e_{ij}]] = [e_{uv}, [e_{ab}, e_{ij}]]_q$$

Proof. 1. By the order definition.

2. It follows from the formula $[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{li} e_{kj}$ in the classical Lie algebra sl_n .

3. Since $e_{ab} < e_{ij}$ we can suppose $b = i$. We have to prove $q^{c_{uv,ab}} [e_{uv}, e_{aj}] = [e_{uv}, e_{aj}]_q$. There are two cases

(a) $e_{uv} < e_{aj}$;

(b) $e_{uv} > e_{aj}$.

(3a): If $[e_{uv}, e_{aj}] \neq 0$ then $v = a$ and $u < v = a < b$, it follows $e_{uv} < e_{ab}$ since $u + v < a + b$. A contradiction. Therefore $[e_{uv}, e_{aj}]_q = [e_{uv}, e_{aj}] = 0$.

(3b): We have to prove

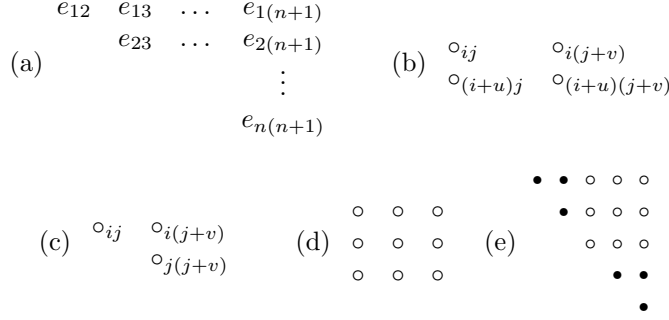
$$q_{uv,ab} [e_{aj}, e_{uv}] = q_{uv,aj} [e_{aj}, e_{uv}]$$

Both sides are zero because if not then $j = u$ and $i < j = u < v$, these imply $i + j < v + u$, and then $e_{ij} < e_{uv}$. Again, we have a contradiction. \square

Theorem X.2. $(sl_{n+1}^+)_q$ is an adequate basic T -Lie algebra.

Proof. Let \mathcal{B} be the canonical basis of sl_{n+1} , and write the basic elements of \mathcal{B} in the form e_{ij} . Now, we put this basic elements in an upper triangular array (Figure 8(a)). Note that, if $\langle e_{ij}, e_{(i+u)(j+v)} \rangle \neq 0$ then the elements appearing in the pseudobacket definition are in a diagonal relationship (Figure 8(b)), and if $[e_{ij}, e_{(i+u)(j+v)}]_q \neq 0$ then $j = i + u$ and we get the Figure 8(c).

So, if we suppose $e_{ij} > e_{uv} > e_{ab}$ then the elements appearing in the formulation of lemma VI.2 (brackets and pseudobrackets) can be fitted inside of a square of the form of Figure 8(d), and such square can be extended to an upper triangle (Figure 8(e)), but this triangle gives a strictly graded algebra of type $(sl_6^+)_q$. Since

FIGURE 8. Some cases in $(sl_{n+1}^+)_q$

these algebras satisfies the condition of lemma VI.2, in particular the elements $e_{ij} > e_{uv} > e_{ab}$ satisfies this condition. Besides,

$$[,]_q((Id \otimes [,]_q)S_{12}S_{23} - ([,]_q \otimes Id)S_{23}S_{12} + (Id \otimes [,]_q)S_{23}S_{12})(e_{ij} \otimes e_{uv} \otimes e_{ab}) =$$

$$q^{c_{uv,ab}+c_{ij,ab}+c_{ij,uv}}([e_{ab}, e_{uv}], e_{ij}] - [e_{ab}, [e_{uv}, e_{ij}]] + [e_{uv}, [e_{ab}, e_{ij}]]) = 0$$

since lemma X.1 and the Jacobi identity in sl_n^+ .

We conclude that $(sl_{n+1}^+)_q$ is an adequate basic T -Lie algebra. \square

Lemma X.3. Suppose $e_{ij} < e_{ab} \in U_q^+(sl_{n+1})$. Then the following equations are satisfied in $U_q^+(sl_{n+1})$,

$$[e_{ij}, e_{ab}] = \begin{cases} e_{ij}e_{ab} - qe_{ab}e_{ij}, & \text{if } i = a \text{ or } j = b \\ e_{ij}e_{ab} - e_{ab}e_{ij} - \langle e_{ij}, e_{ab} \rangle & \text{if } i \neq a, j \neq b \text{ and } j \neq a, \\ e_{ij}e_{ab} - q^{-1}e_{ab}e_{ij}, & \text{if } j = a. \end{cases} \quad (10.24)$$

Proof. By induction on n . For the cases $n = 1, 2, 3, 4, 5$ the equations 10.24 can be verified by straightforward calculations. So we may suppose $n > 5$. Let us consider the Figure 8(a). Such diagram can be thought as formed by two overlapping triangles. The first one, a triangle T_1 with vertices $e_{12}, e_{1n}, e_{(n-1)n}$ and the second one, a triangle T_2 with vertices $e_{23}, e_{2(n+1)}, e_{n(n+1)}$.

The elements in T_i generate a k -subalgebra isomorphic to $U_q^+(sl_n)$, $i = 1, 2$. Then, if e_{ij} and e_{ab} are both in T_1 or T_2 , the equations (10.24) holds. As a consequence, we may suppose $i = 1$ and $b = n + 1$, and put $j \neq n + 1$ and $a \neq 1$.

At the Figure 8(a) join the node rs with the node uv if $[e_{rs}, e_{uv}]_q \neq 0$. We have several cases given by Figure 9, (in the first and third cases, since e_{12}, e_{2j}, e_{nn} are in T_1 and the induction hypothesis there is not arrow between 12 and an , whereas there is not arrow between $2j$ and $n(n + 1)$ because $e_{2j}, e_{an}, e_{n(n+1)}$ are in T_2).

At the first case we get a graph of type A_4 , then $e_{1j} = [e_{12}, e_{2j}]_q$, $e_{a(n+1)} = [e_{an}, e_{n(n+1)}]_q$ are in a subalgebra isomorphic to $U_q^+(sl_5)$, it follows,

$$[e_{1j}, e_{j(n+1)}]_q = e_{1j}e_{j(n+1)} - q^{-1}e_{j(n+1)}e_{1j}$$

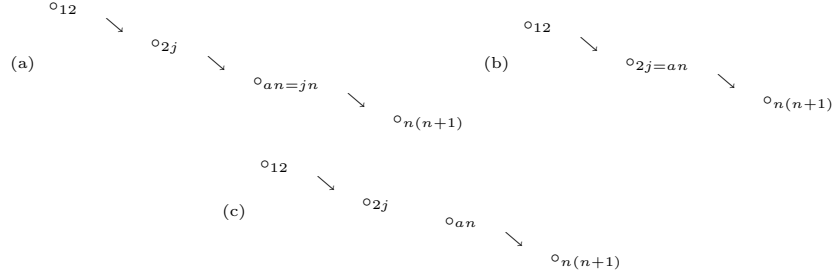


FIGURE 9. (a) First case (b) second case (c) third case.

In the second case we get a graph of type A_3 then $e_{1j} = [e_{12}, e_{2n}]_q$, $e_{2(n+1)} = [e_{2n}, e_{n(n+1)}]_q$ are in a subalgebra isomorphic to $U_q^+(sl_4)$, besides

$$[e_{1j}, e_{a(n+1)}]_q = e_{1n}e_{2(n+1)} - e_{2(n+1)}e_{1n} - (q - q^{-1})e_{1(n+1)}e_{2n}$$

In the third case we may insert the node ja in order to obtain

$$o_{12} \rightarrow o_{2j} \rightarrow o_{ja} \rightarrow o_{an} \rightarrow o_{n(n+1)}$$

this graph is of type A_5 , then $e_{1j} = [e_{12}, e_{2j}]_q$, $e_{a(n+1)} = [e_{an}, e_{n(n+1)}]_q$ are in a subalgebra isomorphic to $U_q(sl_6^+)$ and

$$0 = [e_{1j}, e_{a(n+1)}]_q = e_{1j}e_{a(n+1)} - e_{a(n+1)}e_{1j}$$

Now only remains the cases $e_{1j} = e_{1(n+1)}$, $e_{a(n+1)} = e_{1(n+1)}$. Suppose $e_{1j} = e_{1(n+1)}$. Since $e_{12}e_{a(n+1)} = e_{a(n+1)}$, $e_{2(n+1)}e_{a(n+1)} = qe_{a(n+1)}e_{2(n+1)}$ and $e_{1(n+1)} = e_{12}e_{2(n+1)} - q^{-1}e_{2(n+1)}e_{12}$ it follows,

$$e_{1(n+1)}e_{a(n+1)} = qe_{a(n+1)}e_{1(n+1)}.$$

In a similar way, if $e_{a(n+1)} = e_{1(n+1)}$, we get

$$e_{1j}e_{a(n+1)} = qe_{a(n+1)}e_{1j}.$$

□

Theorem X.4. *There exists an isomorphism*

$$U_q^+(sl_{n+1}) \simeq U(sl_{n+1}^+)_q$$

of k -algebras.

Proof. Let us put $c_{ab,cd} = c_{uv}$ where $x_u = e_{ab}$, $x_v = e_{cd}$, and $x_u < x_v$, $1 \leq a, b, c, d \leq n+1$. From

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$$

it follows, if $e_{ab} < e_{cd}$,

$$c_{ab,cd} = \begin{cases} 1, & \text{if } a = c \text{ or } b = d, \\ 0, & \text{if } a \neq c \text{ and } b \neq d, \\ -1, & \text{if } b = c, \end{cases}$$

Now use lemma X.3 in order to obtain the following equations in $U_q^+(sl_{n+1})$,

$$e_{ab}e_{cd} - q^{c_{ab,cd}}e_{cd}e_{ab} = [e_{ab}, e_{cd}]_q + \langle e_{ab}, e_{cd} \rangle,$$

for all $1 \leq a, b, c, d \leq n+1$.

We conclude $U_q^+(sl_{n+1}) \simeq U(sl_{n+1}^+)_q$. \square

Corollary X.1. 1. *The monomials formed by non-decreasing finite sequences of elements in*

$$\mathcal{B} = \{e_{ij} \mid 1 \leq i < j \leq m\}$$

constitute a free basis of the k -module $U_q^+(sl_{n+1})$, where $m = n(n+1)/2$.

2. *We have*

$$(sl_{n+1}^+)_q|_{t=0} = sl_{n+1}^+$$

and $(sl_{n+1}^+)_q$ is a deformation of sl_{n+1}^+ in the category of T -Lie algebras.

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